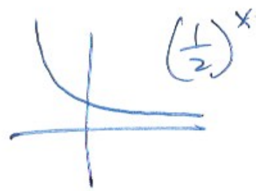
9.3

GEOMETRIC SEQUENCE $a_n = a_1 r^{n-1}$

SERIES $S_n = \underbrace{\dfrac{a_1(r^n - 1)}{r-1}}_{r>1}$ OR $\underbrace{\dfrac{a_1(1-r^n)}{1-r}}_{r<1}$

$$\frac{1}{3} + \frac{1}{6} + \frac{1}{12} + \frac{1}{24} + \cdots$$

$$S_n = \frac{\frac{1}{3}\left(1-\left(\frac{1}{2}\right)^n\right)}{1-\frac{1}{2}} = \frac{2}{3}\left(1-\left(\tfrac{1}{2}\right)^n\right)$$



AS $n \to \infty$, $S = S_\infty \to \frac{2}{3}(1-0) = \frac{2}{3}$

IN GENERAL, GIVEN INFINITE GEOMETRIC SERIES

$$a_1 + a_1 r + a_1 r^2 + a_1 r^3 + \cdots$$

$$S_n = \frac{a_1(1-r^n)}{1-r}$$

IF $r > 1$, EACH TERM IS GREATER THAN PREVIOUS TERM IN SIZE

$S_n \to \infty$ OR $-\infty$

WE SAY THE SERIES DIVERGES

$$S_n = \frac{a_1(1-r^n)}{1-r}$$

$a_1 + a_1 r + a_1 r^2 + a_1 r^3 + \cdots$

IF $r < -1$

eg. $r = -2$

$a_1 - 2a_1 + 4a_1 - 8a_1 + 16a_1 - 32a_1$

$S_1 = a_1$

$S_2 = a_1 - 2a_1 = -a_1$

$S_3 = -a_1 + 4a_1 = 3a_1$

$S_4 = 3a_1 - 8a_1 = -5a_1$

$S_5 = -5a_1 + 16a_1 = 11a_1$

S_n GETS INFINITELY LARGE IN SIZE

SO THE SERIES DIVERGES

IF $|r| < 1$

eg. $r = \frac{2}{3}$

$r^n = \left(\frac{2}{3}\right)^n$

$\left\{\frac{2}{3}, \frac{4}{9}, \frac{8}{27}, \frac{16}{81}, \frac{32}{243}, \frac{64}{729}, \ldots\right\}$

$r^n \to 0$

AS $n \to \infty$, $S_n \to \frac{a_1(1-0)}{1-r} = \frac{a_1}{1-r}$

eg. $r = -\frac{2}{3}$

$\left\{-\frac{2}{3}, \frac{4}{9}, -\frac{8}{27}, \frac{16}{81}, -\frac{32}{243}, \frac{64}{729}, \ldots\right\}$

$r^n \to 0$

AS $n \to \infty$, $S_n \to \frac{a_1}{1-r}$

eg. $\frac{1}{3} + \frac{1}{6} + \frac{1}{12} + \frac{1}{24} + \cdots$ SERIES CONVERGES TO $\frac{2}{3}$

$*\frac{1}{2}$ $*\frac{1}{2}$ $*\frac{1}{2}$

$$S = \frac{\frac{1}{3}}{1-\frac{1}{2}} = \frac{\frac{1}{3}}{\frac{1}{2}} = \frac{2}{3}$$

SO, IF $|r| < 1$ IE. $-1 < r < 1$

$$a_1 + a_1 r + a_1 r^2 + \cdots = \frac{a_1}{1-r}$$

IE. SERIES CONVERGES TO $\frac{a_1}{1-r}$

IF $r = 1$, $a_1 + a_1 + a_1 + a_1 + \cdots \longrightarrow$
- ∞ IF $a_1 > 0$ (IE. SERIES DIVERGES)
- 0 IF $a_1 = 0$ (IE. SERIES CONVERGES TO 0)
- $-\infty$ IF $a_1 < 0$ (IE. SERIES DIVERGES)

IF $r = -1$

$$\underbrace{a_1 - a_1}_{=0} + \underbrace{a_1 - a_1}_{=0} + \underbrace{a_1 - a_1}_{=0} + \cdots \to 0$$

$$a_1 \underbrace{- a_1 + a_1}_{=0} \underbrace{- a_1 + a_1}_{=0} \underbrace{- a_1 + a_1}_{=0} - \cdots \to a_1$$

IF $a_1 = 0$, SERIES CONVERGES TO 0

IF $a_1 \neq 0$, SERIES DIVERGES

$$a_1 - a_1 + a_1 - a_1 + \cdots$$

$S_1 = a_1$

$S_2 = a_1 - a_1 = 0$

$S_3 = 0 + a_1 = a_1$

$S_4 = a_1 + a_1 = 0$

S_n DOES NOT HEAD TO ANY 1 NUMBER EXCEPT IF $a_1 = 0$

INFINITE GEOMETRIC SERIES CONVERGES TO

$$S = \frac{a_1}{1-r}$$

FOR $|r| < 1$;

IF $|r| \geq 1$

THE SERIES DIVERGES

WRITE $0.6\overline{54}$ AS A FRACTION IN SIMPLEST TERMS.

$$= 0.6545454545454\ldots$$

$$= 0.6 + \left[\underbrace{0.054 + 0.00054 + 0.0000054 + \cdots}_{\text{INFINITE GEOMETRIC SERIES}}\right]$$

$0.054 \xrightarrow{*\frac{1}{100}} 0.00054 \xrightarrow{*\frac{1}{100}} 0.0000054$

$a_1 = 0.054 = \frac{54}{1000}$

$r = \frac{1}{100}$

$$= \frac{6}{10} + \frac{\frac{54}{1000}}{1 - \frac{1}{100}} \cdot \frac{1000}{1000}$$

$$= \frac{6}{10} + \frac{54}{1000 - 10}$$

$$= \frac{\cancel{6}^{3}}{\cancel{10}_{5}} + \frac{\cancel{54}\,\cancel{6}\,3}{\cancel{990}\,\cancel{110}\,55}$$

$$= \frac{3}{5} + \frac{3}{55}$$

$$= \frac{33 + 3}{55}$$

$$= \frac{36}{55}$$

ON JAN 1, 2008, JOE DEPOSITS \$2,000 INTO A RETIREMENT ACCOUNT

HIS FUNDS GROW IN THE ACCOUNT BY 5% PER YEAR

IN ADDITION, ON EVERY JAN 1 AFTER THAT, HE DEPOSITS ANOTHER \$2,000.

HOW MUCH IS HIS ACCOUNT WORTH ON JAN 1, 2048?

a_n = AMOUNT IN THE ACCOUNT ON JAN 1 OF THE n-TH YEAR

a_1 = AMOUNT ON JAN 1 OF YEAR 1 (2008) = \$2000

a_2 = " " " 2 (2009)

$$= \underbrace{2000 + 2000 * 0.05}_{\text{PREVIOUS FUNDS + GROWTH}} + \underbrace{2000}_{\text{NEW DEPOSIT}}$$

$$= 2000(1+0.05)$$

$$= 2000\underbrace{(1.05)^1}_{100\%+5\%=105\%} + 2000$$

$$a_3 = [2000(1.05)+2000](1.05) + 2000$$

$$= 2000(1.05)^2 + 2000(1.05) + 2000$$

← GEOMETRIC SERIES (WRITTEN IN REVERSE)

$a_1 = 2000$

$r = 1.05$

$$a_n = \underbrace{2000(1.05)^{n-1} + 2000(1.05)^{n-2} + \cdots + 2000}_{n \text{ TERMS}}$$